Data-driven modelling of turbulent flows

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Data-driven modelling

- How to **identify flow models** from **data**?

- Flow behavior is typically complex,
  data are **hard to interpret** directly.

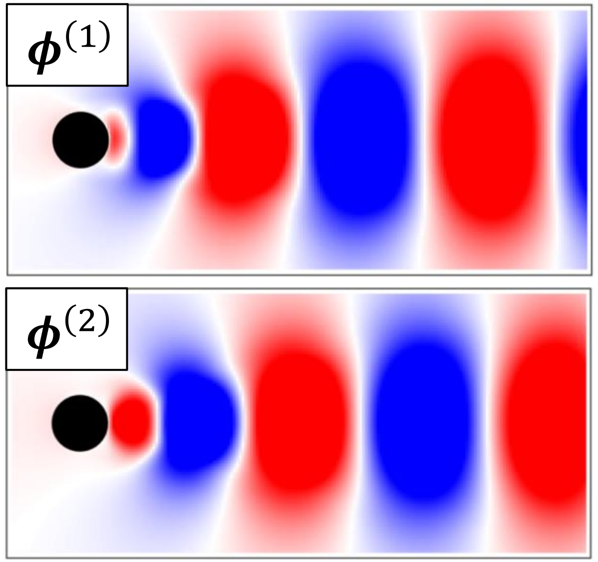
- ***Divide and conquer*** strategy:
  - model the flow as a superposition of simpler modes;
  - understand the interaction between modes.

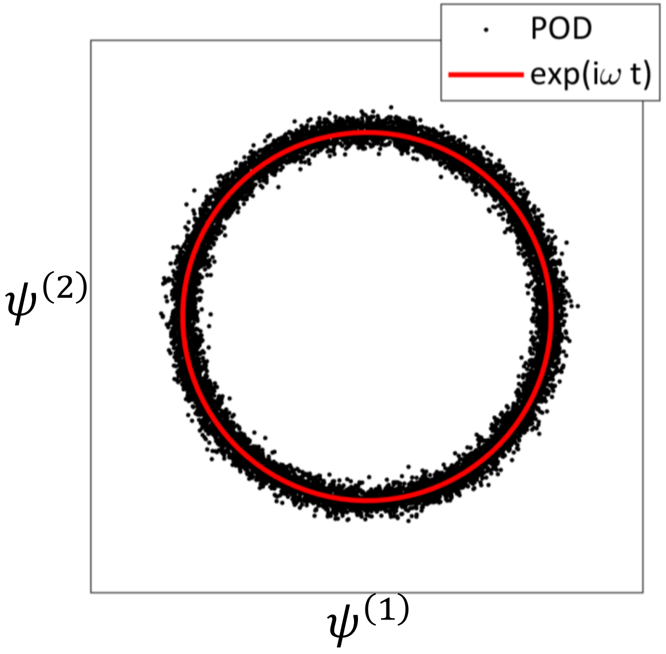
Proper Orthogonal Decomposition

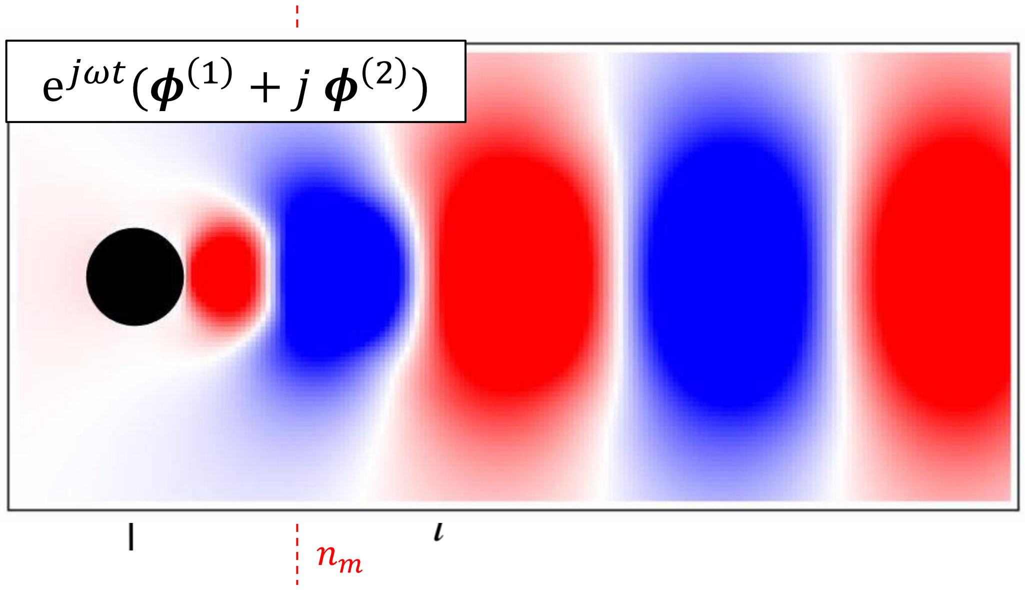
- In its most common implementation, POD separate variables in a vector field:

$$\boldsymbol{a}(\boldsymbol{x},t) = \sum_{i=1}^{r} \psi^{(i)}(t)\, \sigma^{(i)}\, \boldsymbol{\phi}^{(i)}(\boldsymbol{x})$$

- $\boldsymbol{\phi}^{(i)}(\boldsymbol{x})$ are the spatial basis functions
- $\psi^{(i)}(t)$ are the temporal basis function
- $\left|\sigma^{(i)}\right|^2$ are representative of an "energy" (kinetic energy for a velocity field)







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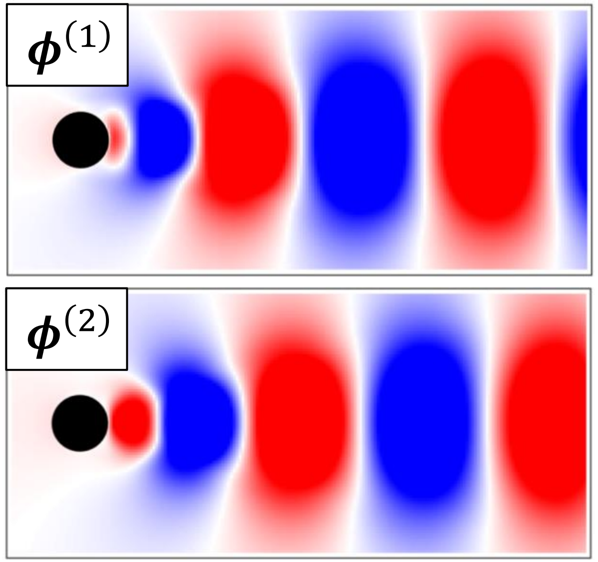
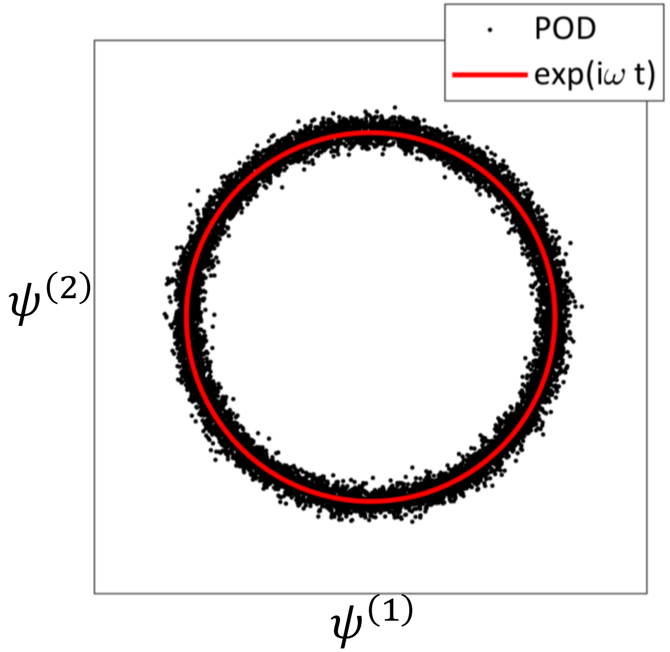
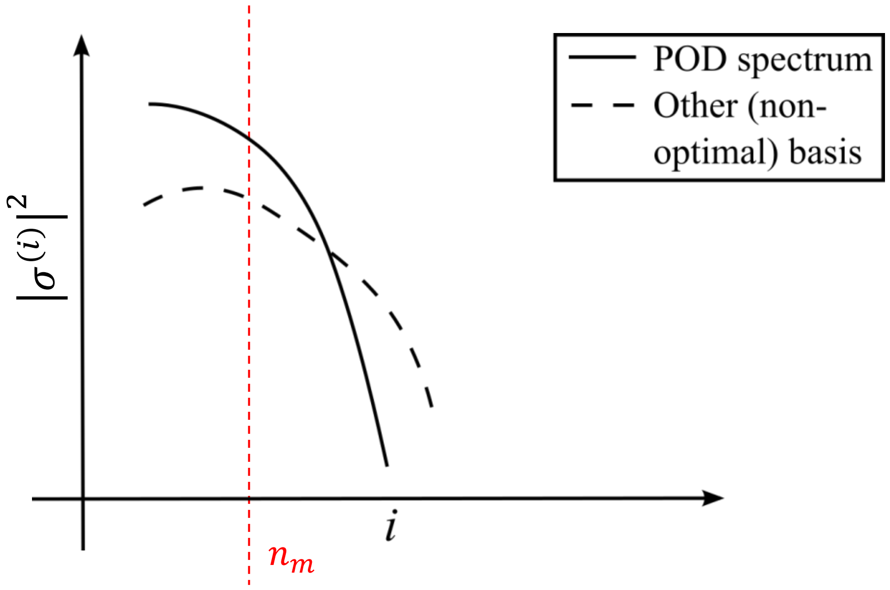
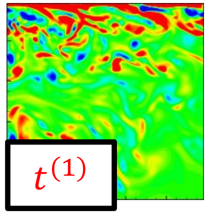




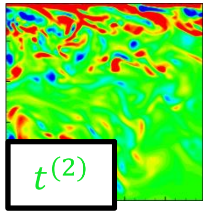


POD in discrete space

- Snapshot method (Sirovich, 1987):

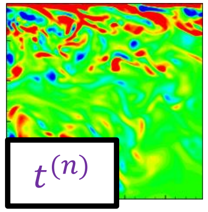


$= a\left(\boldsymbol{x}, t^{(1)}\right)$



$= a\left(\boldsymbol{x}, t^{(2)}\right)$

⋮



$= a\left(\boldsymbol{x}, t^{(n)}\right)$

$$A = \begin{bmatrix} a\left(\boldsymbol{x}, t^{(1)}\right) \\ a\left(\boldsymbol{x}, t^{(2)}\right) \\ \vdots \\ a\left(\boldsymbol{x}, t^{(n)}\right) \end{bmatrix}$$

- Time correlation matrix:

$$\mathrm{R_t} = \mathrm{AA^T} = \Psi\Sigma\Sigma^T\Psi^\mathrm{T}$$

$$\Psi = \begin{bmatrix} \psi^{(1)}\left(t^{(1)}\right) & \cdots & \psi^{(r)}\left(t^{(1)}\right) \\ \vdots & \ddots & \vdots \\ \psi^{(1)}\left(t^{(n_t)}\right) & \cdots & \psi^{(r)}\left(t^{(n_t)}\right) \end{bmatrix}$$

- Spatial correlation matrix:

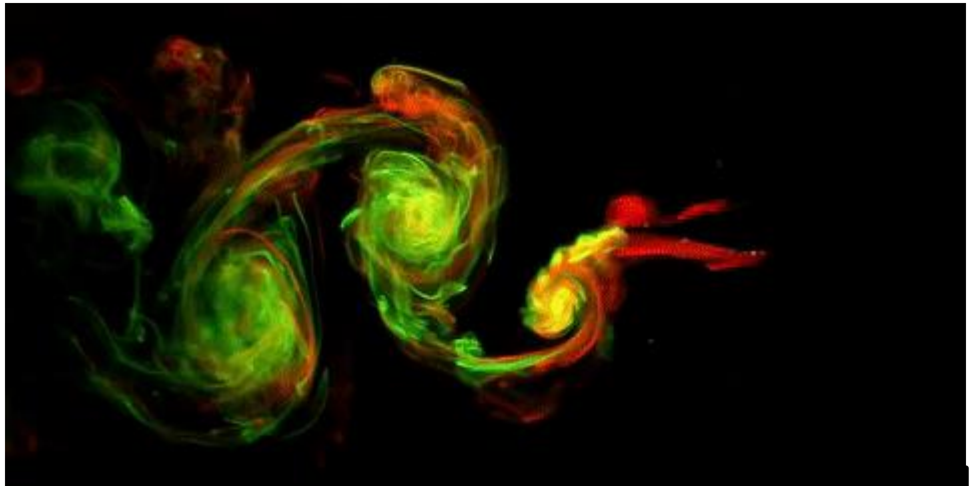
$$\mathrm{R_s} = \mathrm{A^T A} = \Phi\Sigma^\mathrm{T}\Sigma\Phi^\mathrm{T}$$

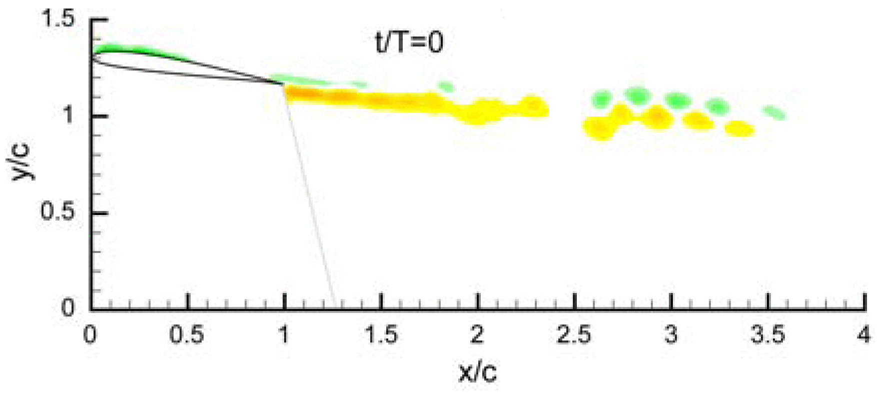
$$\Phi = \begin{bmatrix} \phi^{(1)}\left(\underline{x}^{(1)}\right) & \cdots & \phi^{(r)}\left(\underline{x}^{(1)}\right) \\ \vdots & \ddots & \vdots \\ \phi^{(1)}\left(\underline{x}^{(n_p)}\right) & \cdots & \phi^{(r)}\left(\underline{x}^{(n_p)}\right) \end{bmatrix}$$

- Singular Value Decomposition (SVD):

$$\mathrm{A} = \Psi\Sigma\Phi^\mathrm{T}$$

Unsteady forces on flapping wings


Mitchel et al. (2015) APS Gallery of Fluid Motion



Rival et al. (2009), Exp Fluids

- Flapping wings are characterized by vortices over the wing due to flow separation.
- These vortices produces low pressures, resulting in high lift and propulsive forces.
- Existing data-driven force models cannot be easily interpreted.
- Is it possible to identify an interpretable data-driven model for the forces?
- Using a flow-based decomposition to identify the relation between vortices and forces on the wing.

Experimental setup

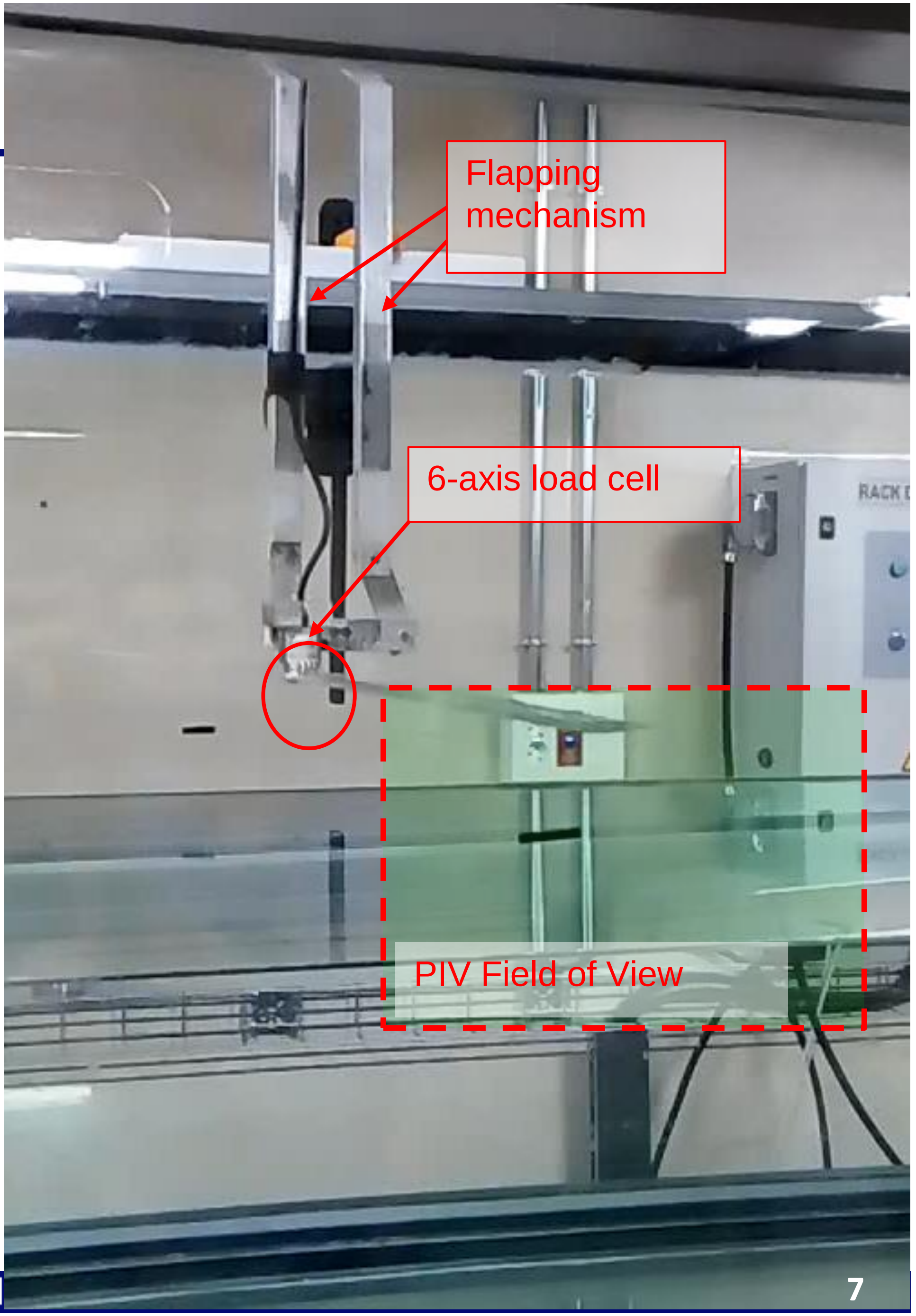
- Flapping kinematics:

$$h(t) = c\,\sin(2\pi f t)$$

$$\theta(t) = \theta_0 \sin\left(2\pi f t + \frac{\pi}{2}\right)$$

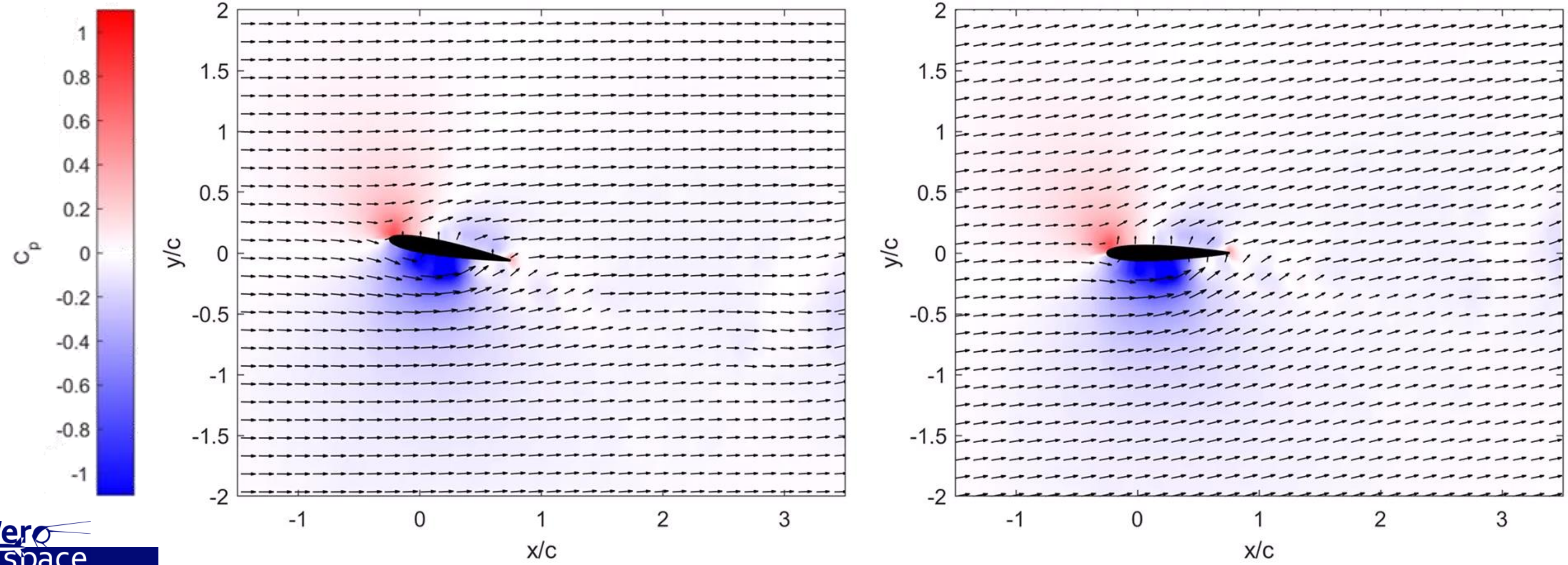
$St = \frac{2cf}{V_\infty} = 0.2$
$Re = \frac{\rho V_\infty c}{\mu} = 3600$
$k = \frac{\pi f c}{V_\infty} = 0.63$
$\theta_0 = 10°$

- Measurements:
  - 80 phase-averaged 2D-PIV velocity fields in the midspan section;
  - Time-resolved aerodynamic loads from the load-cell.



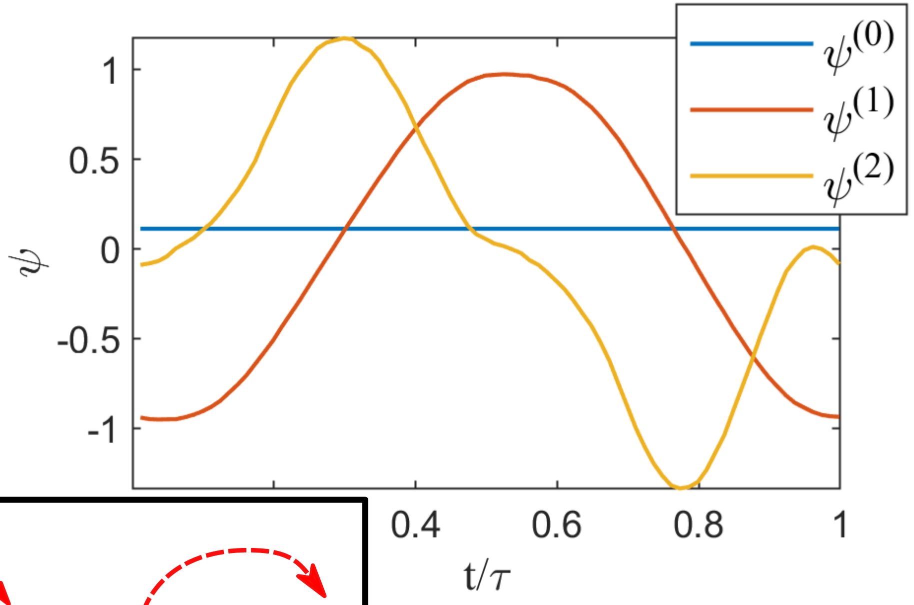
Changing the reference frame

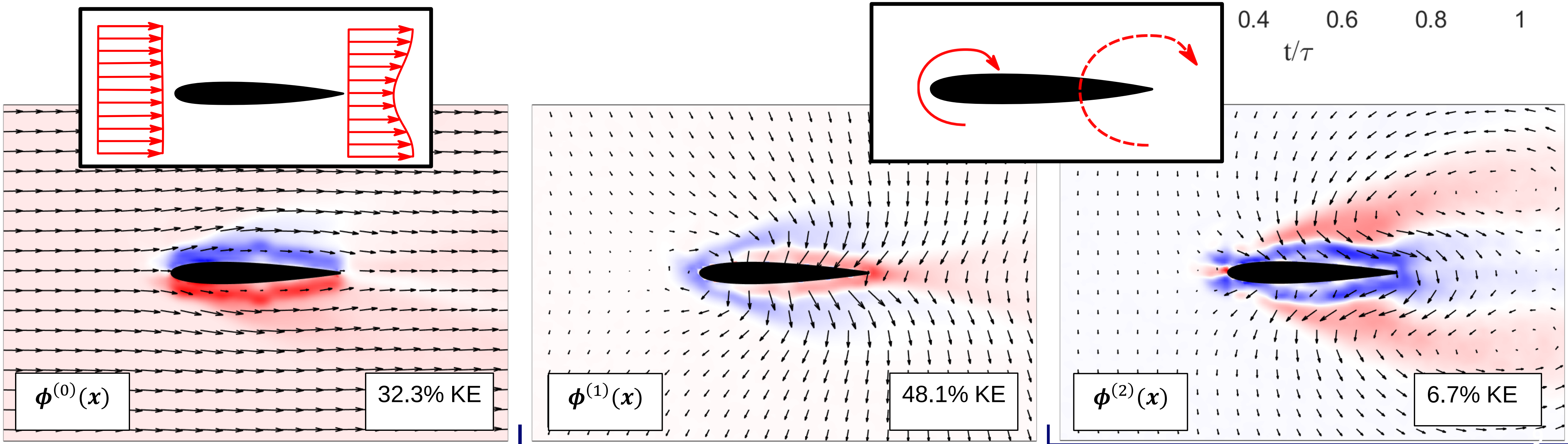
- The fluid domain changes with time.
- POD does not directly account for time-varying domain.
- The reference frame has been re-centered on the wing to avoid changes in the domain boundaries.



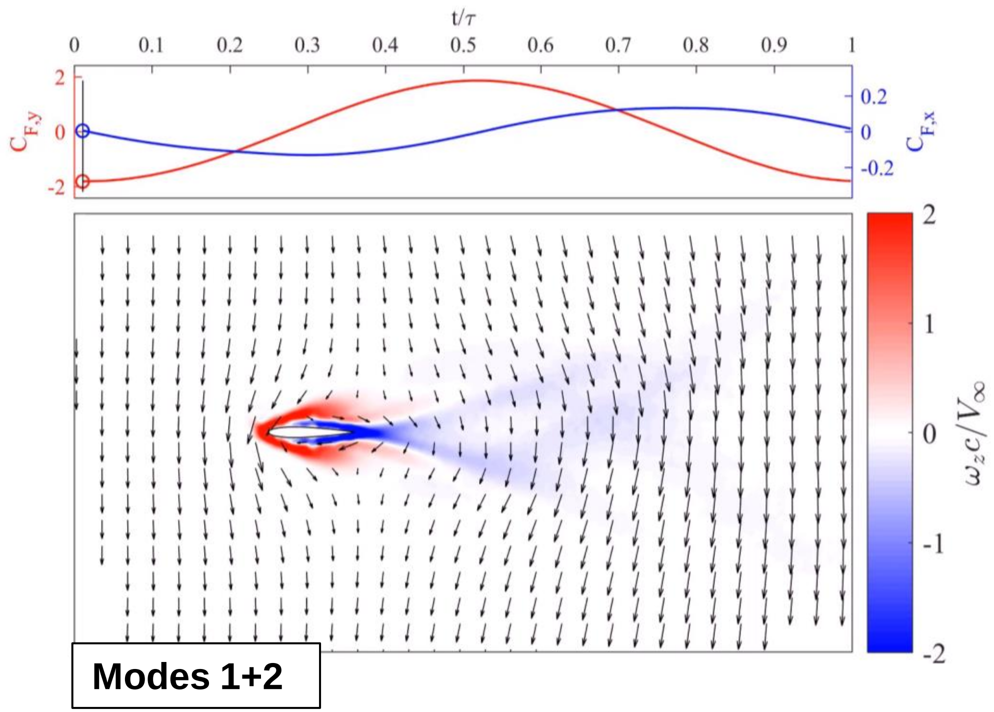
Velocity Decomposition: results

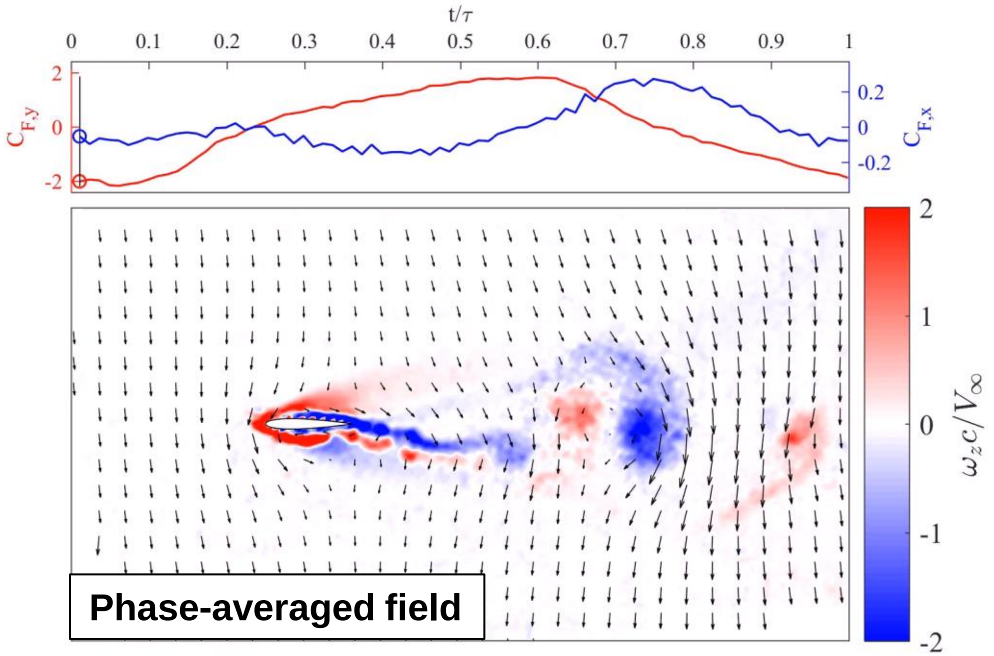
- Mode 0 is the time average.
- It represent a flow parallel to the chord.
- Modes 1+2 are “sinusoidal” contributions in phase quadrature.
- They represent the circulation over the airfoil.





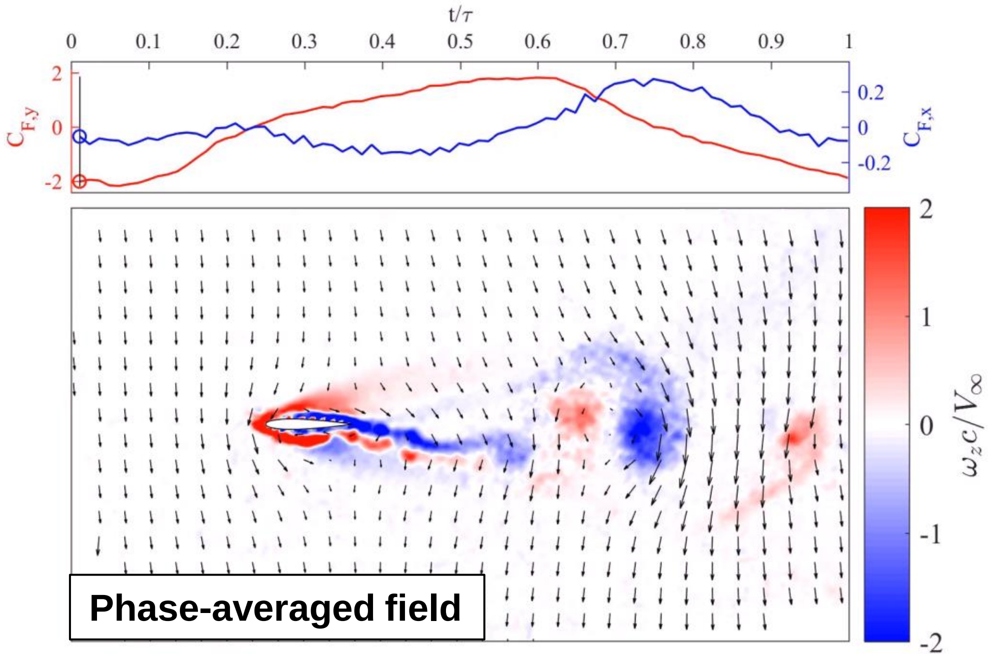
Low-order model

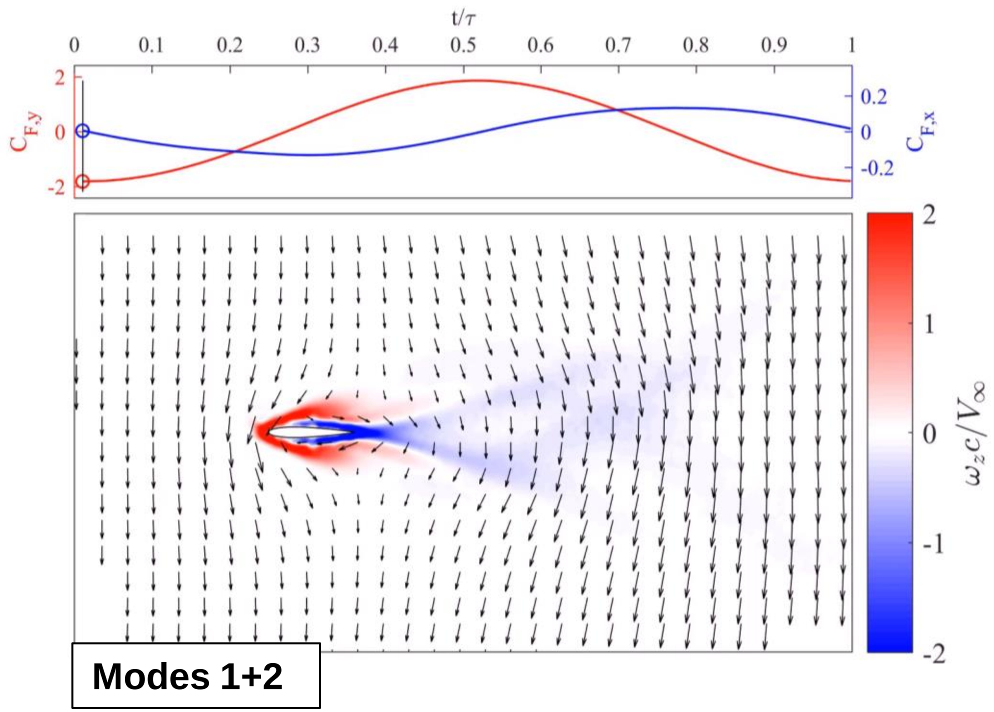


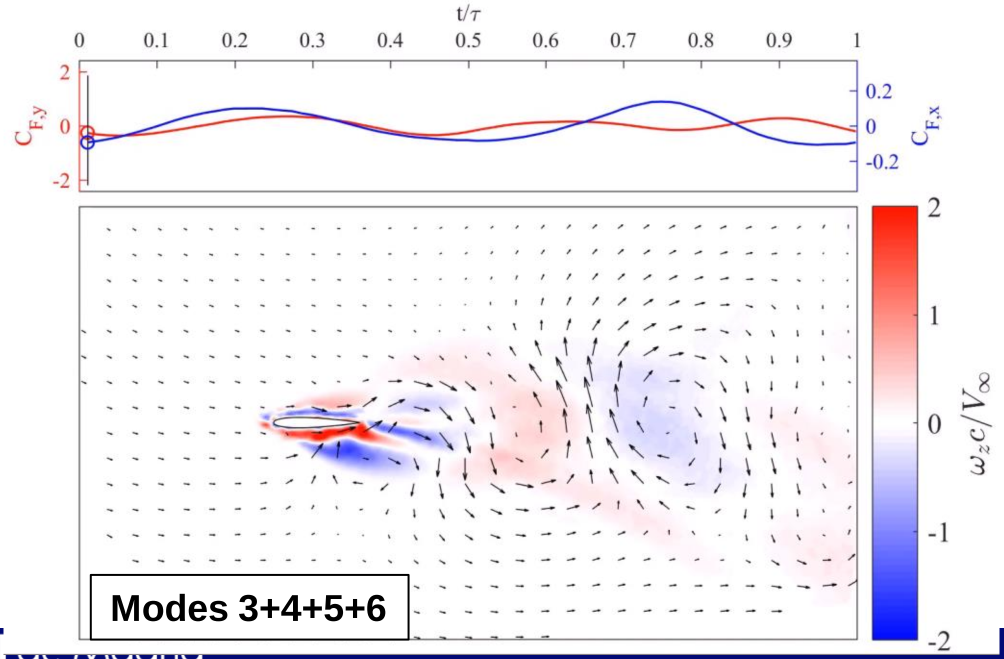


$$\sum_{i=1}^{2} \psi^{(i)}\sigma^{(i)}\phi^{(i)} \approx \mathrm{e}^{-j\omega t}\left(\sigma^{(1)}\phi^{(1)} + j\,\sigma^{(2)}\phi^{(2)}\right)$$

Low-order model







POD and Stochastic Estimation

- **Linear Stochastic Estimation (LSE)**
  given the multipoint signals a(x,t) and b(x,t), their linear relation X is given (stochastically) by

$$(A^T A)\,\mathrm{X} = B^T A \quad \Rightarrow \quad \mathrm{X} = (A^T A)^{-1} B^T A$$

$$A = \begin{bmatrix} a\left(\boldsymbol{x}_a, t^{(1)}\right) \\ \vdots \\ a\left(\boldsymbol{x}_a, t^{(n_t)}\right) \end{bmatrix}$$

$$B = \begin{bmatrix} b\left(\boldsymbol{x}_b, t^{(1)}\right) \\ \vdots \\ b\left(\boldsymbol{x}_b, t^{(n_t)}\right) \end{bmatrix}$$

- **POD**

$$A = \Psi_A \Sigma_A \Phi_A^T \quad \Rightarrow \quad \Psi_A = A \Phi_A \Sigma_A^{-1}$$

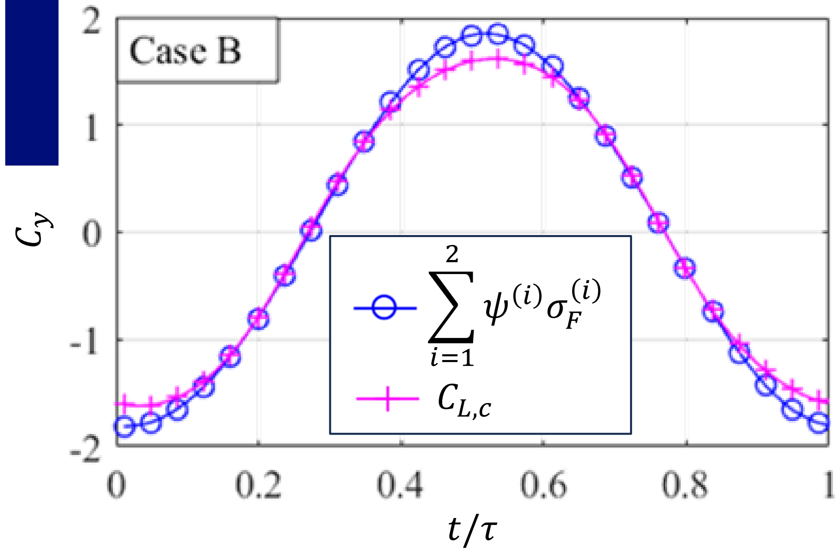
Ψ: temporal basis

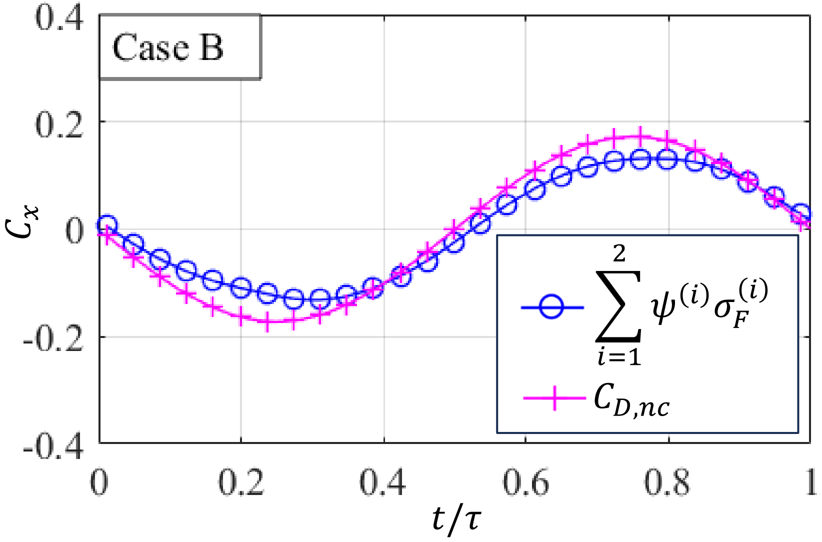
Φ: spatial basis

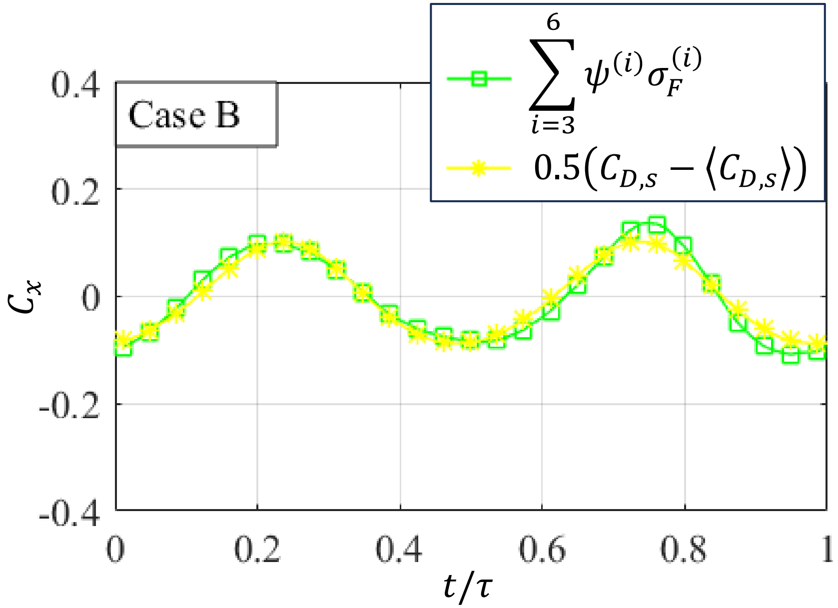
- **POD-LSE (or Extented POD, Borée, 2003)**
  the LSE of the temporal POD modes provides the stochastic linear relation

$$(\Psi_A^T \Psi_A)\,(\Sigma_B \Phi_B^T) = B^T \Psi_A \quad \Rightarrow \quad \Sigma_B \Phi_B^T = B^T \Psi_A$$

Flow Field / Force Model: results







Modes 1,2:
circulation on the wing and the rotational acceleration of the frame

circulatory y-force: $^{*}C_{L,c}(t) = 2\pi C(k)Q(t)$

$$Q(t) = \alpha_{eff}(t) - \theta_m + \frac{c}{2V_\infty}\dot{\theta}(t)(1.5 - 2\xi_p)$$

non-circulatory x-force: $^{*}C_{D,nc} = \frac{\pi c}{2V_\infty}\dot{\alpha}(t)$

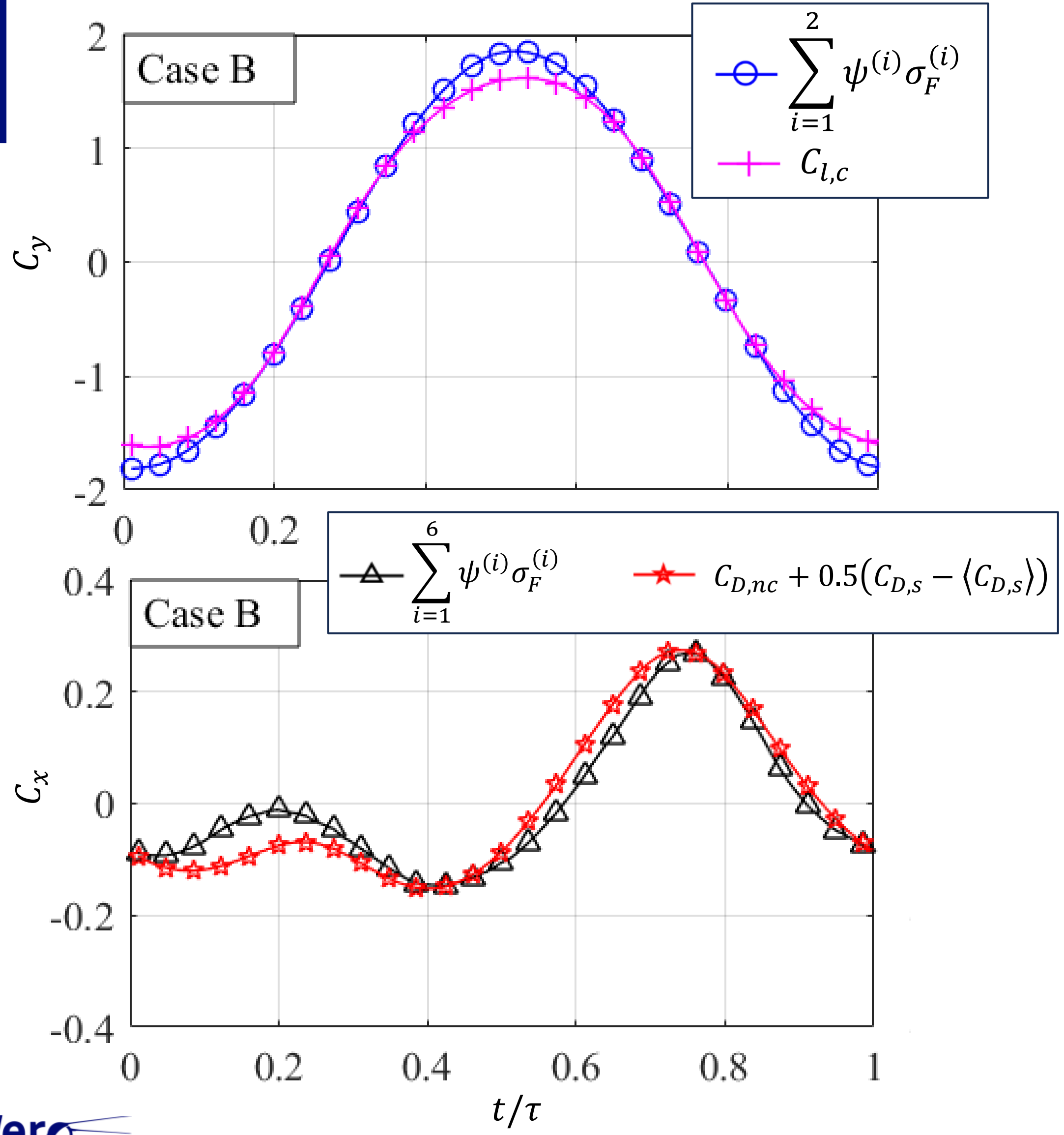
Modes 3,4,5,6:
wake shedding

suction x-force: $^{\dagger}C_{D,s} = -2\pi\left(2C(k)Q(t) - \frac{c}{2V_\infty}\dot{\alpha}\right)^2$

*** Theodorsen, T. (1935) NACA report 496**

† Garrick, I.E. (1937) NACA report 567

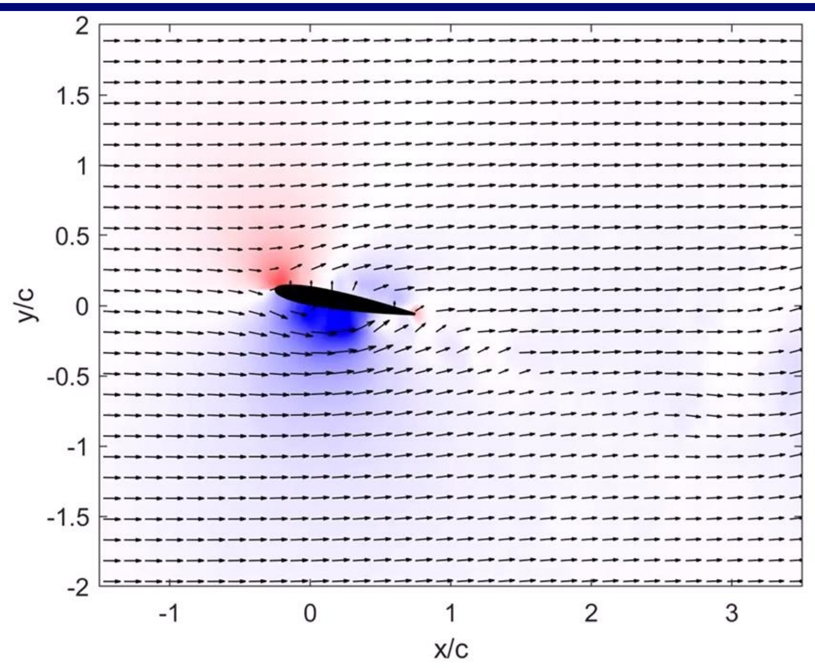
Flow Field / Force Model: summary

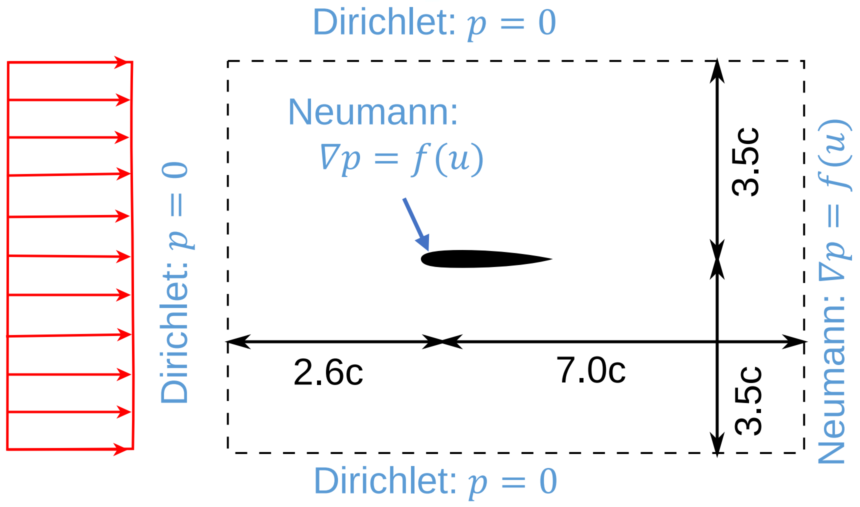


- Using **POD** and **LSE** a model linking flow fields and forces is extracted from data*.
- The model is physically sound:
  - Modes 1 and 2 model wing circulation and reference frame rotation and provide circulatory force on y and added-mass force on x.
  - Higher order modes model wake shedding and provide suction force on x.
- The link is **purely stochastic**!
- Can we use a **more deterministic approach**?

***Raiola et al, ETFS, 2021**

Pressure computation





- Pressure and velocity are linked through the Poisson equation:

$$\nabla^2 p = -\rho \nabla \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$$

  - Finite differences solver;
  - Neumann boundary condition on the airfoil

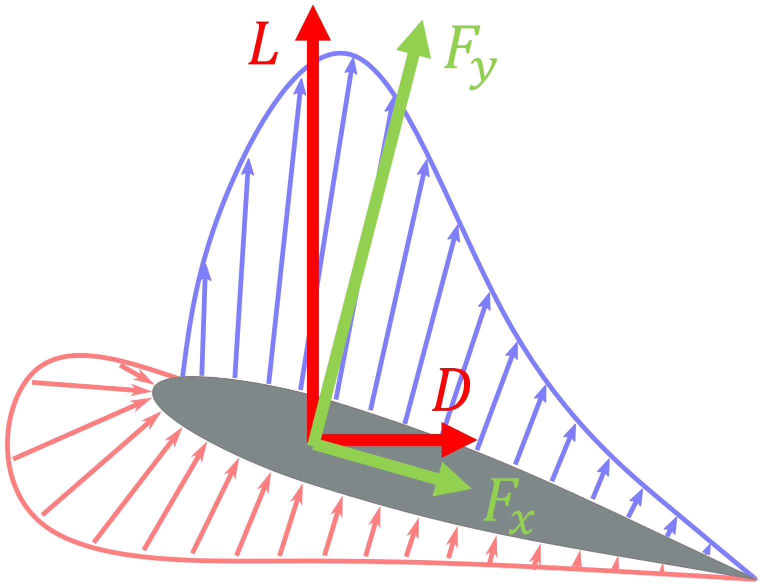
$$\nabla p = -\rho \frac{D\boldsymbol{u}}{Dt} - \mu \nabla^2 \boldsymbol{u}$$

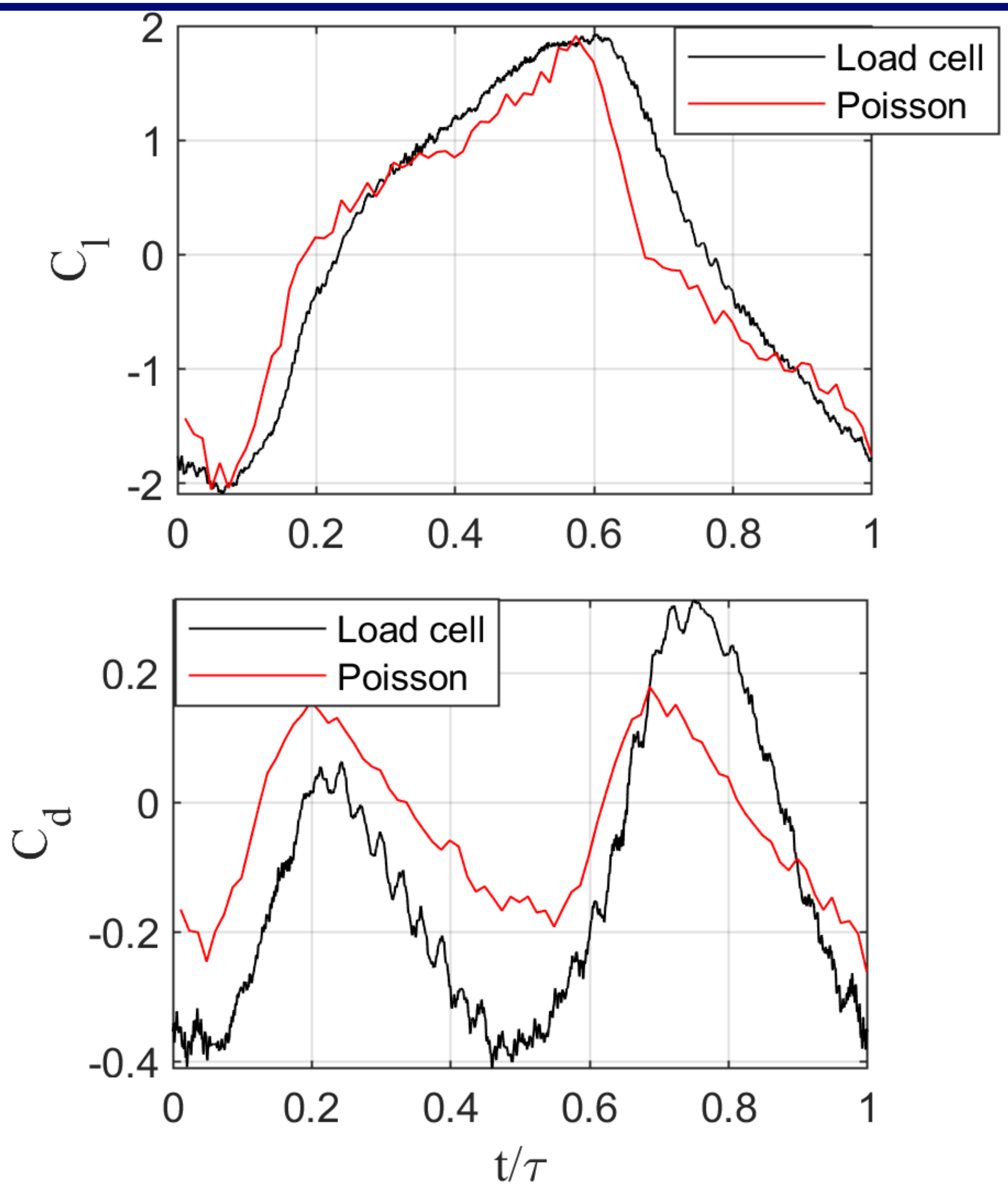
  - Neumann and Dirichlet conditions on the external boundaries;
  - velocity fields are interpolated on a finer mesh
    317×244 vect. $\rightarrow$ 634×479 vect.

Loads computation

- Loads are computed from the pressure over the airfoil surface:

$$\boldsymbol{F} = \oint p\, \boldsymbol{n}\, ds$$





Velocity/Pressure Decomposition

- Performing a **Galerkin projection** of the Poisson equation:

$$\left.\begin{aligned} \nabla^2 p &= -\rho\nabla\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \\ \boldsymbol{u} &= \sum_{i=0}^{n}\psi^{(i)}\sigma^{(i)}\boldsymbol{\phi}^{(i)} \end{aligned}\right\} \Rightarrow \nabla^2 p(\boldsymbol{x},t) = \sum_{i=0}^{n}\sum_{j=0}^{n}\underbrace{\psi^{(i)}\psi^{(j)}}\sigma_P^{(i,j)}\nabla^2 P^{(i,j)}$$

$$\nabla^2 P^{(i,j)}(\boldsymbol{x}) = -\frac{\sigma^{(i)}\sigma^{(j)}}{\sigma_P^{(i,j)}}\rho\nabla\cdot\left(\boldsymbol{\phi}^{(i)}(\boldsymbol{x})\cdot\nabla\right)\boldsymbol{\phi}^{(j)}(\boldsymbol{x})$$

Quadratic relation!

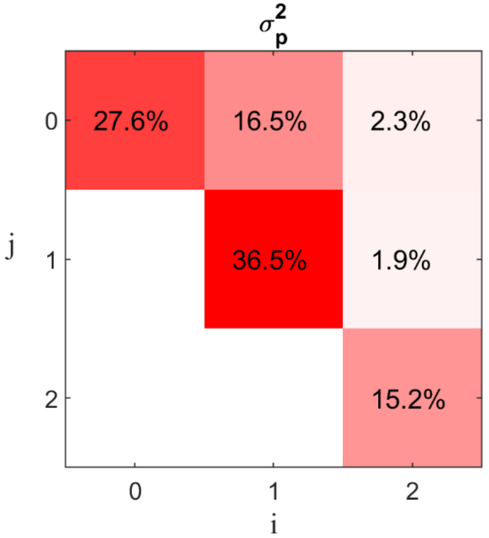
$\psi^{(i)}$:	Time Mode i
$\sigma^{(i)}$:	Singular Value i
$\boldsymbol{\phi}^{(i)}$:	Space Mode i
$P^{(i,j)}$:	Pressure contribution of modes i and j

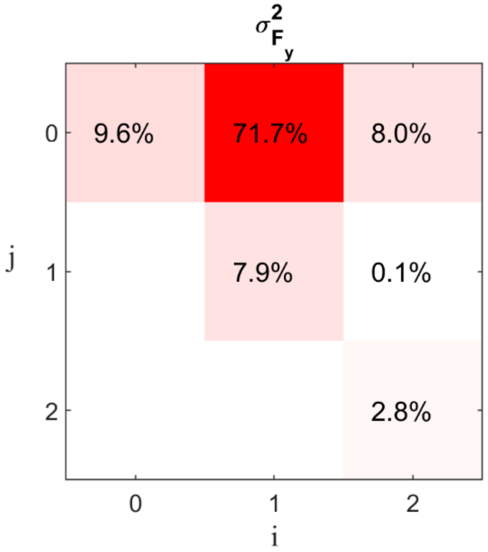
- The quadratic relation between velocity modes and pressure can be retrieved using the **Quadratic Stochastic Estimation (QSE)**:

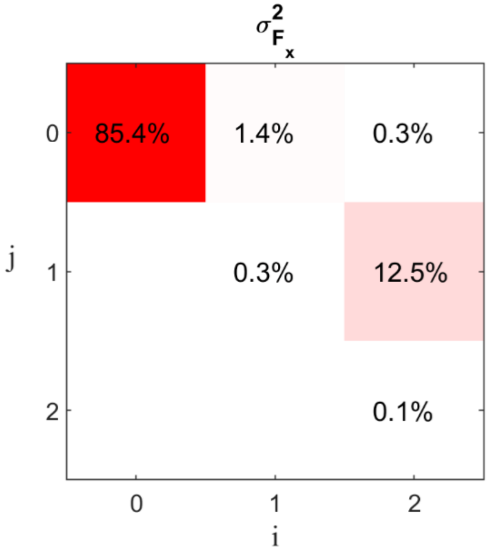
$$(\Upsilon^T\Upsilon)\,\mathrm{P} = p^T\Upsilon \Rightarrow \quad \mathrm{P} = (\Upsilon^T\Upsilon)^{-1}p^T\Upsilon$$

$$\Upsilon = \begin{bmatrix} \psi^{(0)}(t_1)\psi^{(0)}(t_1) & \cdots & \psi^{(0)}(t_1)\psi^{(n)}(t_1) & \cdots & \psi^{(n)}(t_1)\psi^{(n)}(t_1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \psi^{(0)}(t_n)\psi^{(0)}(t_n) & \cdots & \psi^{(0)}(t_n)\psi^{(n)}(t_n) & \cdots & \psi^{(n)}(t_n)\psi^{(n)}(t_n) \end{bmatrix}$$

Velocity/Pressure Decomposition: results







- The QSE is performed on the first 3 POD modes:

$$p(\boldsymbol{x},t) \approx \sum_{i=0}^{3}\sum_{j=0}^{3} \psi^{(i)}(t)\psi^{(j)}(t)\sigma_P^{(i,j)}P^{(i,j)}(\boldsymbol{x})$$

$$F_y(t) \approx \sum_{i=0}^{3}\sum_{j=0}^{3} \psi^{(i)}(t)\psi^{(j)}(t)\sigma_{F_y}^{(i,j)}$$

$$F_x(t) \approx \sum_{i=0}^{3}\sum_{j=0}^{3} \psi^{(i)}(t)\psi^{(j)}(t)\sigma_{F_x}^{(i,j)}$$

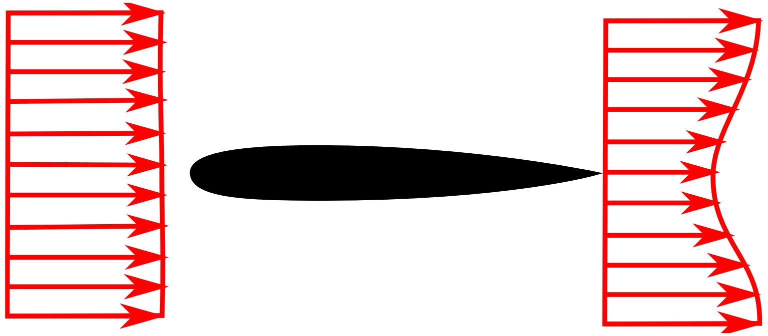
- Most of the terms are relevant for the pressure field (σ_P).
- The normal force is mainly dominated by linear contributions $P^{(0,0)}$, $P^{(0,1)}$, $P^{(0,2)}$.
- The chordwise force is mainly dominated by $P^{(0,0)}$and $P^{(1,2)}$.

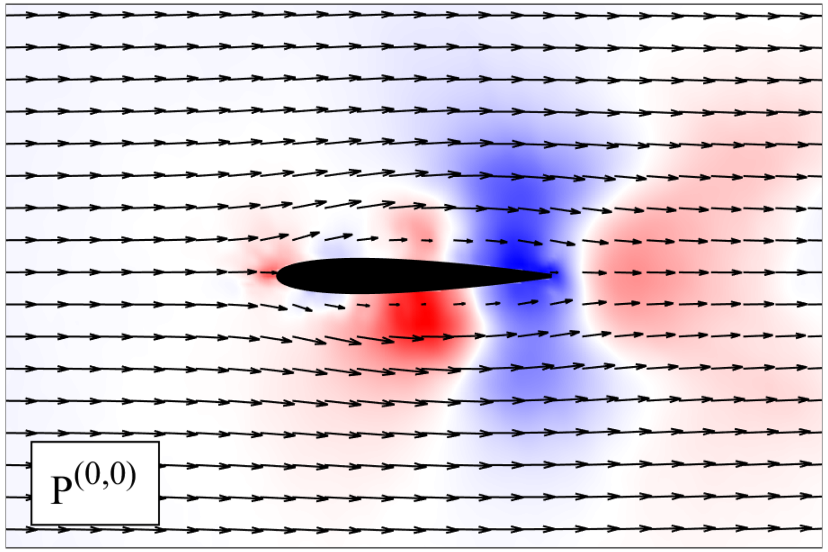
Velocity/Pressure Decomposition

- The contribution $P^{(0,0)}$ is constant in time:

$$\sigma_P^{(0,0)} P^{(0,0)}$$

- Contribution of the chord-wise flow.
- Pressure acts mainly in the chord-wise direction.

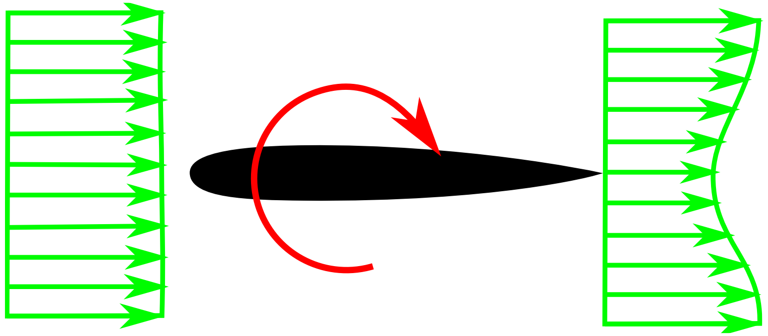




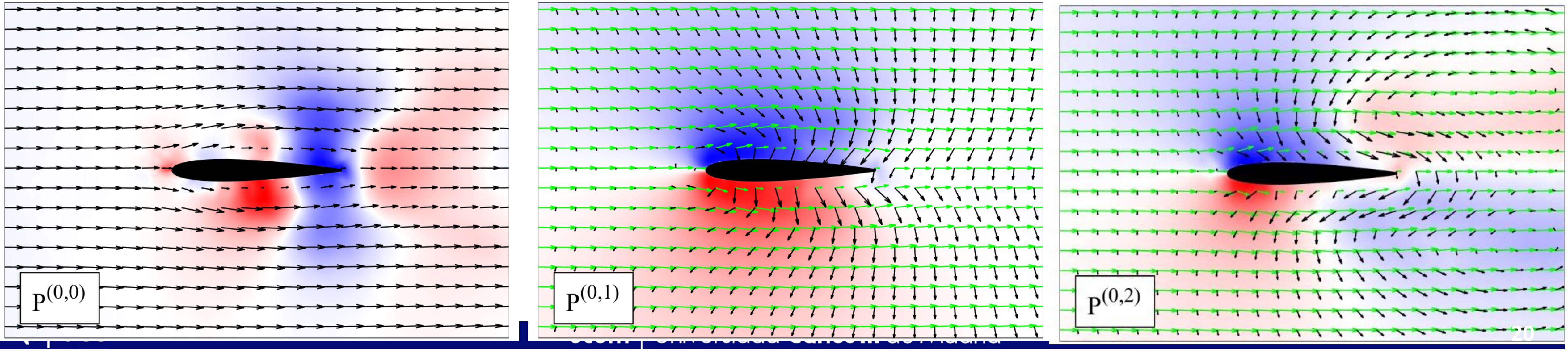
Velocity/Pressure Decomposition

- The contributions $P^{(0,1)}$ and $P^{(0,2)}$ are linear:

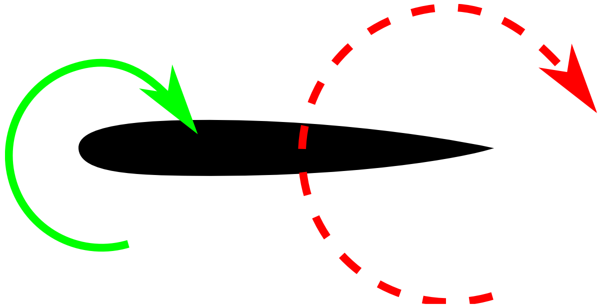
$$\psi^{(i)}(t)\sigma_P^{(0,i)}P^{(0,i)}$$



- Interaction between the circulation (modes 1+2) and the chordwise flow in mode 0.
- This pressure is antisymmetric wrt the chord.
- It is associated mainly with chord-normal forces.



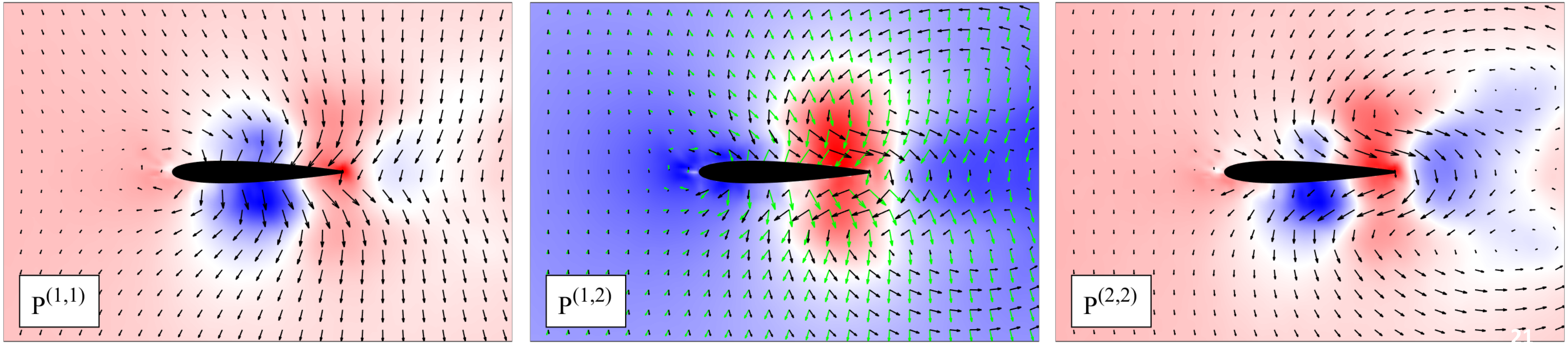
Velocity/Pressure Decomposition



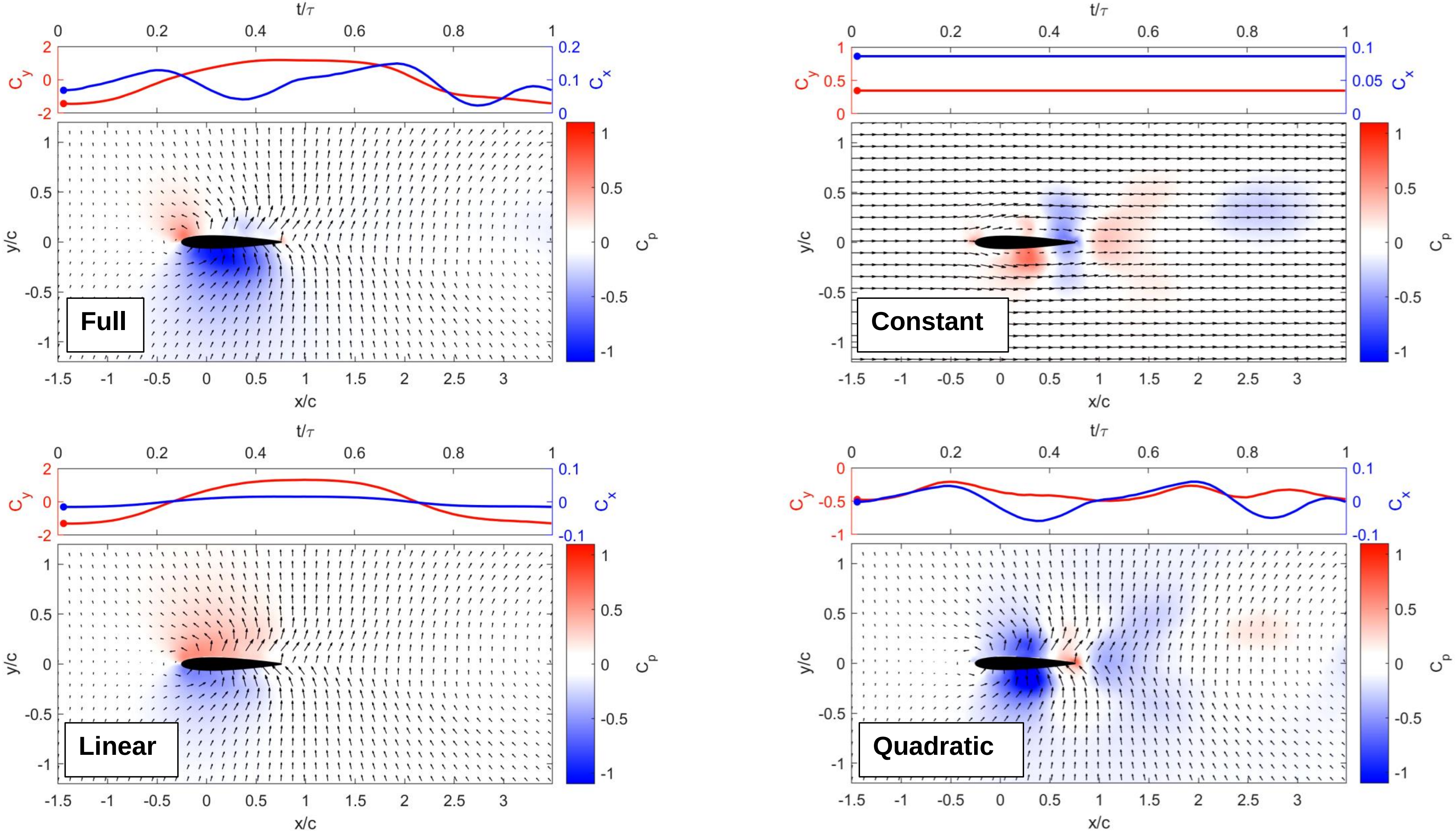
- The contributions $P^{(1,1)}$, $P^{(1,2)}$ and $P^{(2,2)}$ are quadratic:

$$\psi^{(i)}(t)\psi^{(j)}(t)\sigma_P^{(i,j)}P^{(i,j)}$$

- Mutual interaction between the circulation (modes 1+2).
- It mainly contribute to chordwise force.



Reduced model: velocity and pressure fields



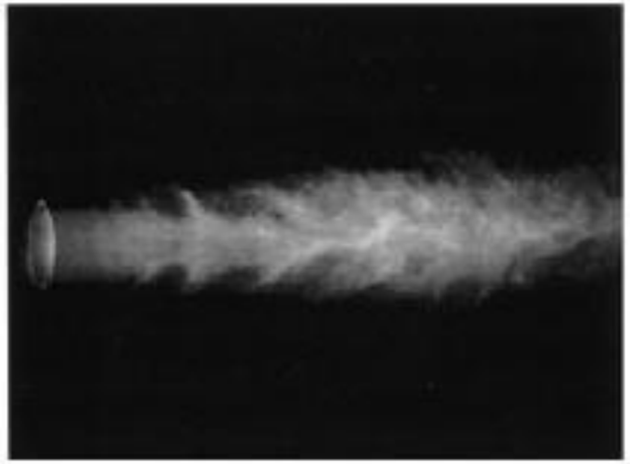
Velocity/pressure decomposition: summary

- **POD** and **QSE** can provide a combined velocity/pressure decomposition of the flow features on the wing*.
- The main flow features are:
  - chord-wise mean flow;
  - time-evolving vortex over the wing.
- A **more deterministic force/flow field model** is obtained:
  - a constant chord-wise force contribution from the chord-wise mean;
  - a linear chord-normal force contribution arising from the interaction between the chordwise flow and the wing vortex
  - a quadratic chord-wise force contribution from the vortex.

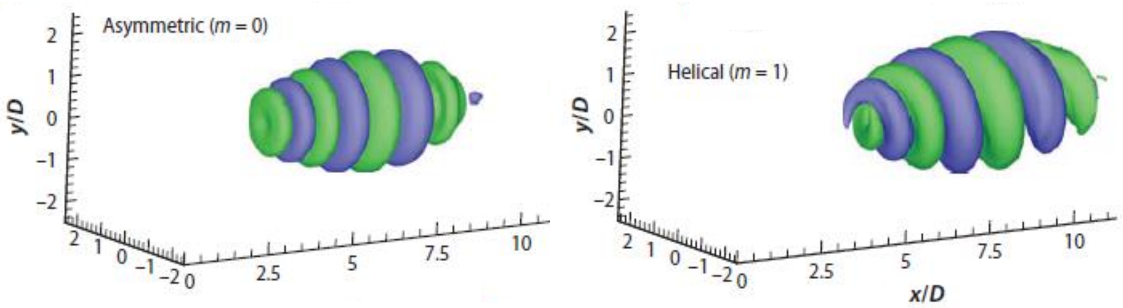
*Raiola, LXLASER2022, 2022

Advecting wavepackets in subsonic jet noise

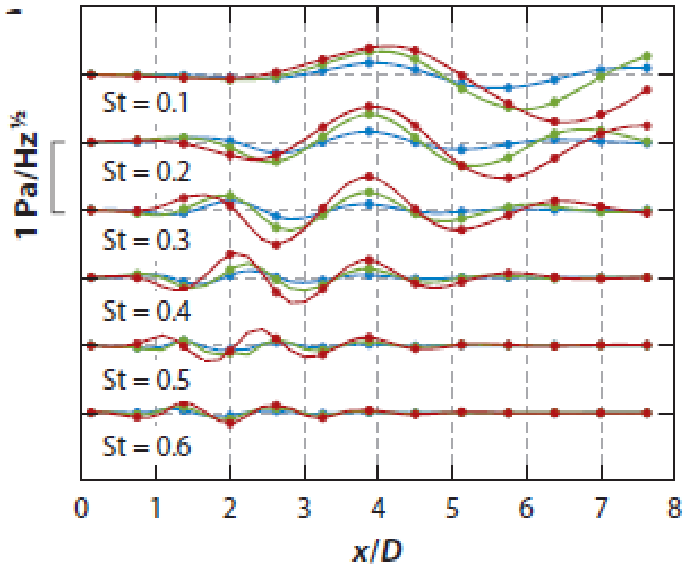
- Subsonic jet noise is dominated by sound emission from convective flow structures in the jet.
- These structures are referred to as **wavepackets**.
- These structures are:
  - coherent in the azimutal direction;
  - modulated in the axial direction;
  - temporally intermittent;
  - not highly energetic.



Crow & Champagne, JFM, 1971



Freund, JFM, 2001



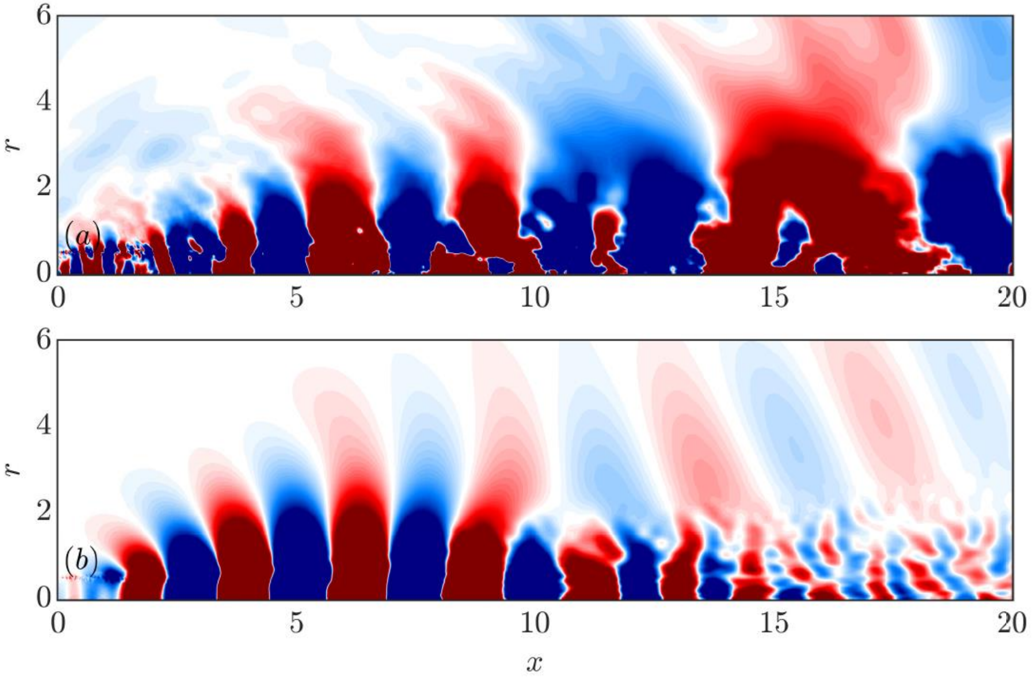
Suzuki & Colonius, JFM, 2006

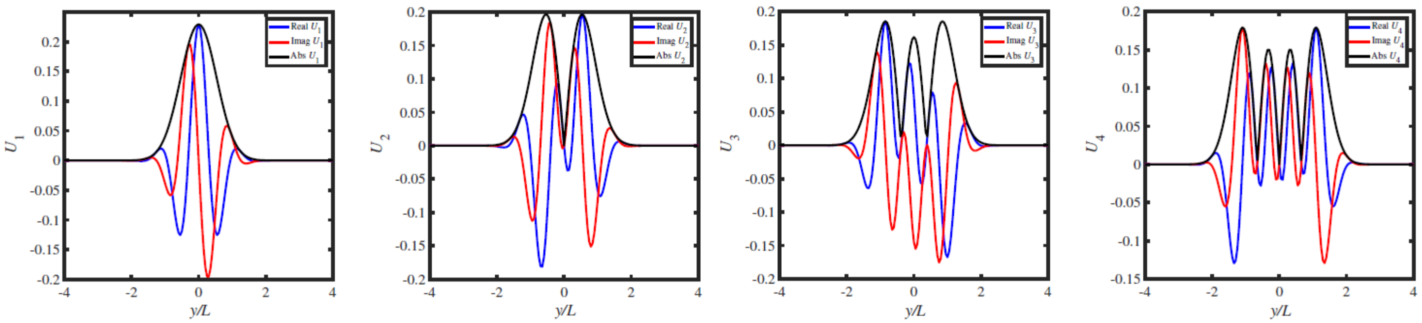
Identification of wavepackets

- A large body of work is dedicated to *educe* wavepackets from jet flows.
- Wavepackets should be coherent in the spatio-temporal sense:

$$A(x,t)e^{j(kx-\omega t)}$$

- SPOD (Towne et al., 2017) is generally used to detect and study wave-packets.
- The main drawback of SPOD is the need for **time resolution**.
- The successful application of this technique is mostly limited to LES data, limiting the number of studies in this subject.





(a) Mode 1 (b) Mode 2 (c) Mode 3 (d) Mode 4

Cavalieri et al., AMR, 2019

Hilbert Transform

- A travelling wave in the complex domain is given by:

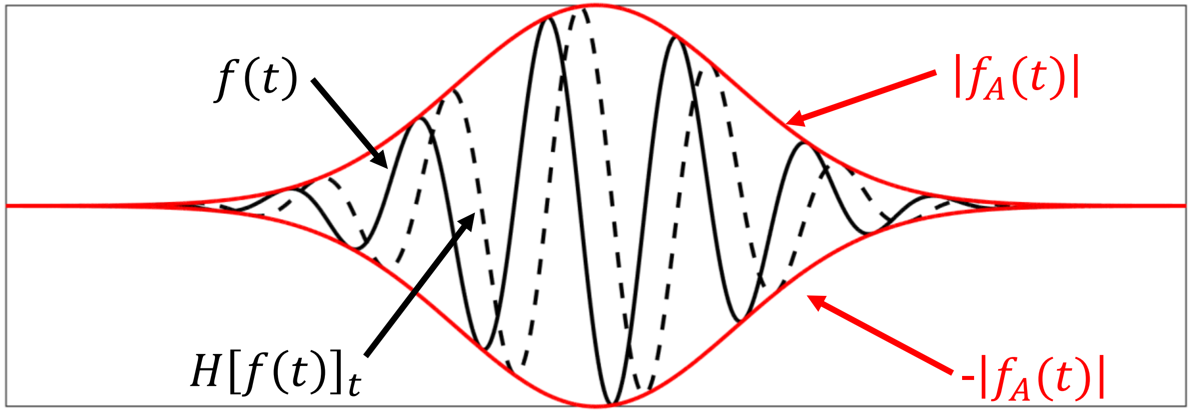
$$(a+jb)e^{-j\omega t} = \underbrace{(a\cos(\omega t) + b\sin(\omega t))}_{f(t)} + j\underbrace{(b\cos(\omega t) - a\sin(\omega t))}_{\mathrm{H}[f(t)]_t}$$

- The Hilbert Transform H produce a $\pi/2$ shift of the signal:

$$\mathrm{H}[f(t)]_t = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{f(\tau)}{(t-\tau)} d\tau$$

- The complex-valued extension of $f(t)$ is the Analytic Signal $\hat{f}(t)$

$$\hat{f}(t) = f(t) + j\,H[f(t)]_t = |f_A(t)|\,e^{j\theta(t)}$$



Hilbert POD

- Using the Analytic Signal (in time) and standard SVD, a complex extension of the POD* can be obtained:

$$\hat{\boldsymbol{u}}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + jH[\boldsymbol{u}(\boldsymbol{x},t)]_t = \sum_i \hat{\psi}_i(t)\ \sigma_i\ \hat{\boldsymbol{\phi}}_i(\boldsymbol{x})$$

Complex Time Mode — Real Singular Value — Complex Space Mode

- **Time resolution is required**!
- Is it possible to trade **space resolution** for **time resolution**?
- For travelling modes:

$$\sum_i |\hat{\psi}_i(t)|\ \sigma_i\ |\hat{\boldsymbol{\phi}}_i(\boldsymbol{x})|\, e^{j(\mathrm{kx}-\omega t)} = \boldsymbol{u}(\boldsymbol{x},t) + jH[\boldsymbol{u}(\boldsymbol{x},t)]_x = \hat{\boldsymbol{u}}(\boldsymbol{x},t)$$

$$\mathrm{H}[u(x,t)]_x = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{u(\lambda,t)}{(x-\lambda)} d\lambda$$

- The Hilbert transform can be performed also in **space** *!

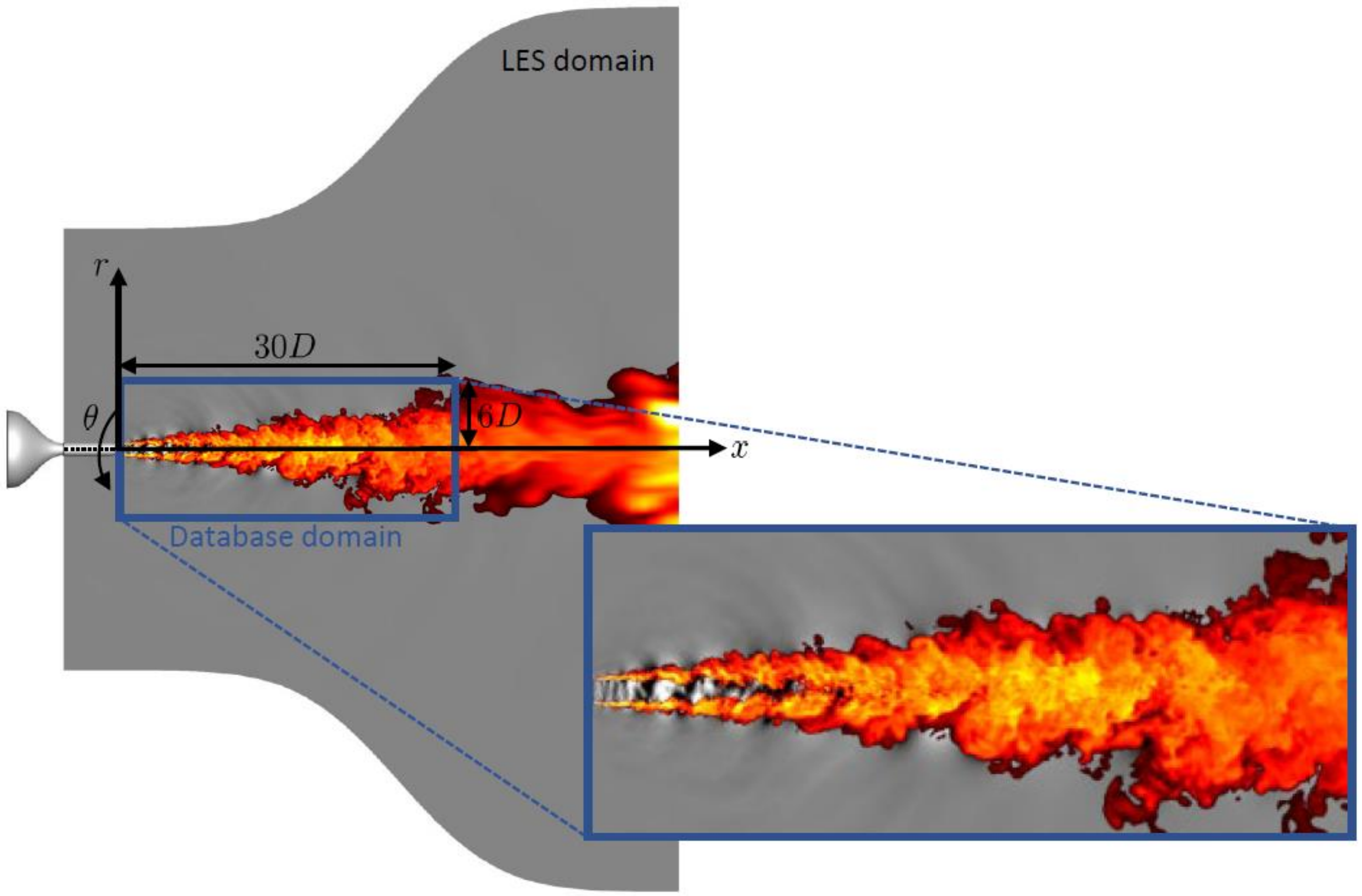
***Barnett, MWR, 1982; Kriegseis et al., ISPIV 2021**

***Raiola, SFMC, 2022**

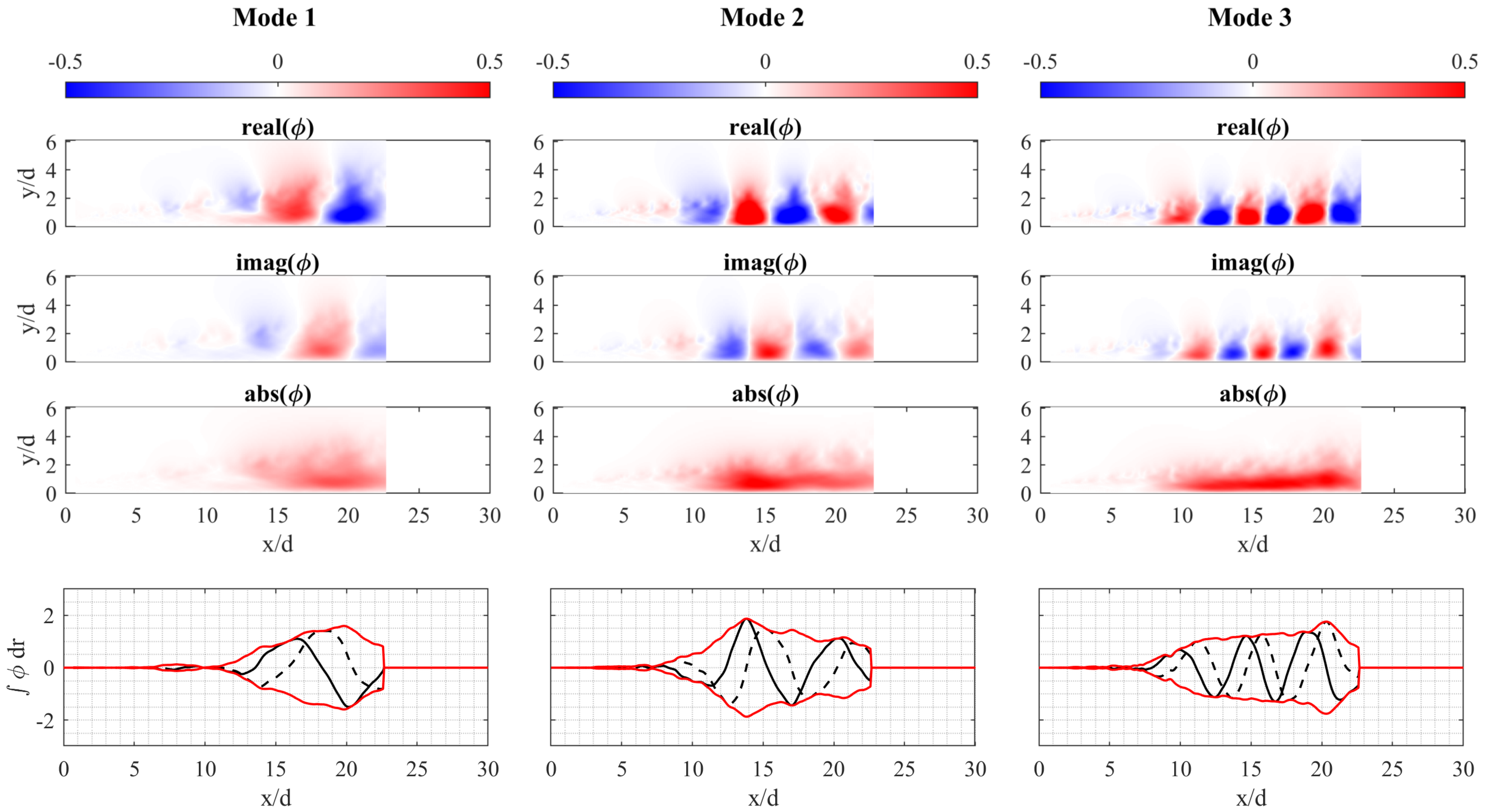
Turbulent Jet Dataset

- High Fidelity LES dataset (Towne et al, AIAA, 2023):
  - $M_{jet} = 0.9$
  - $Re_{jet} = \frac{U_{jet}\, d}{\nu} \approx 10^6$
  - $\Delta t = 0.2 \frac{d}{c_0}$
  - 10000 snapshots
  - $30d \times 6d$ in x-r plane
  - $656 \times 138 \times 128$ gridpoints

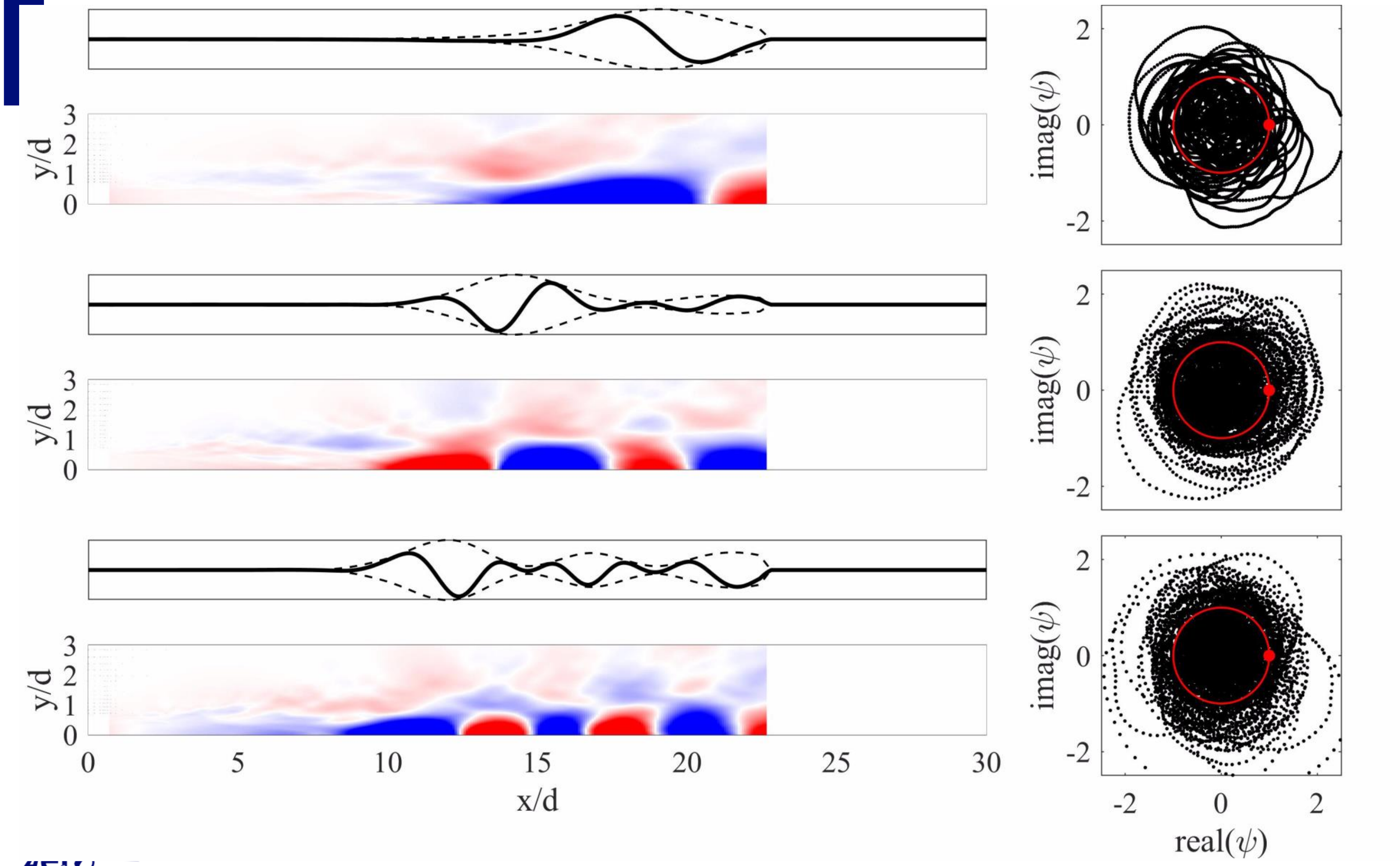
- Only the axisymmetric part is used, i.e. azimuthal wavenumber 0.
- Streamwise and radial velocity fields in a x-r plane.



HPOD results



Oscillator model

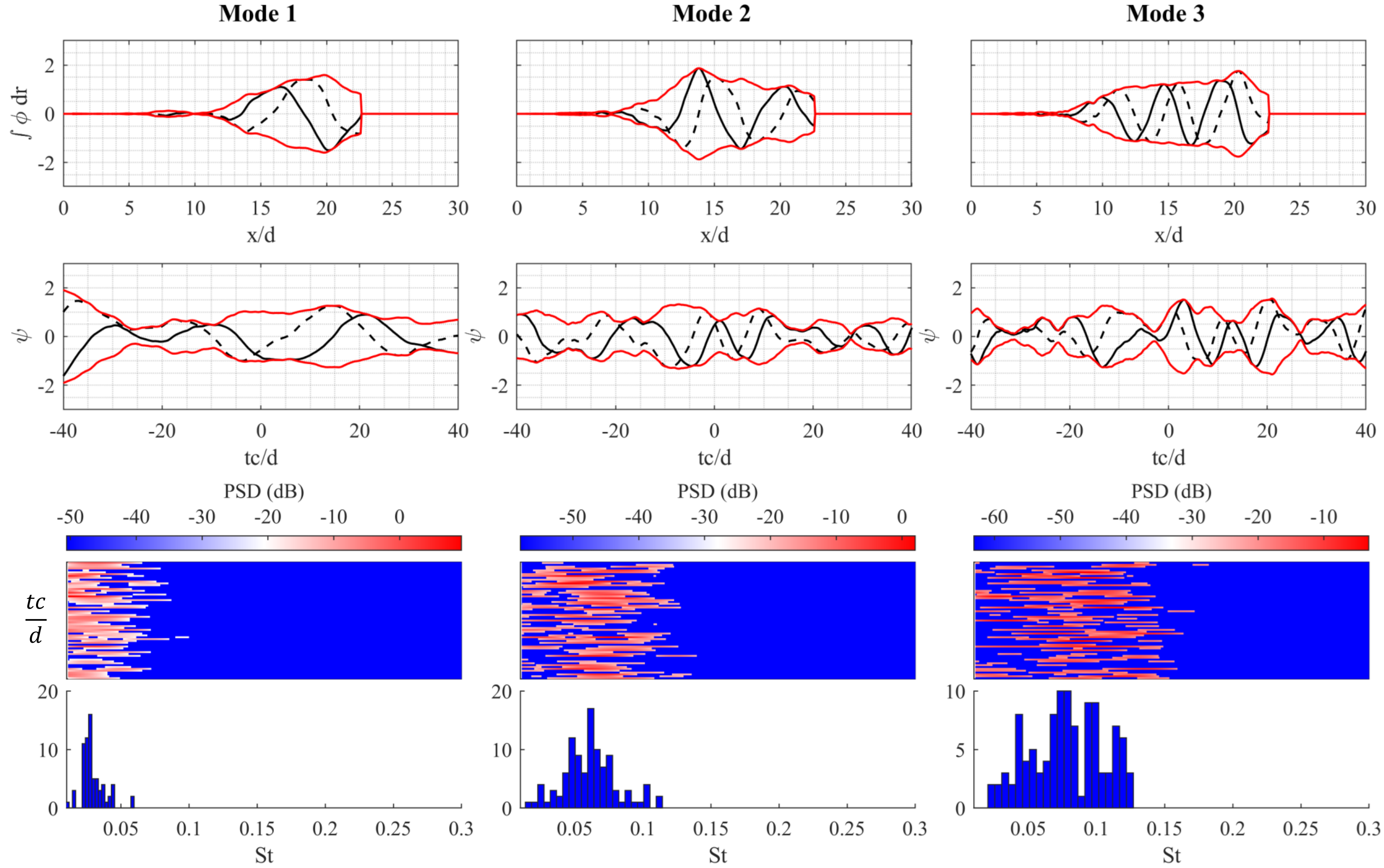


- The wavepacket behavior is better highlighted through an oscillator model:

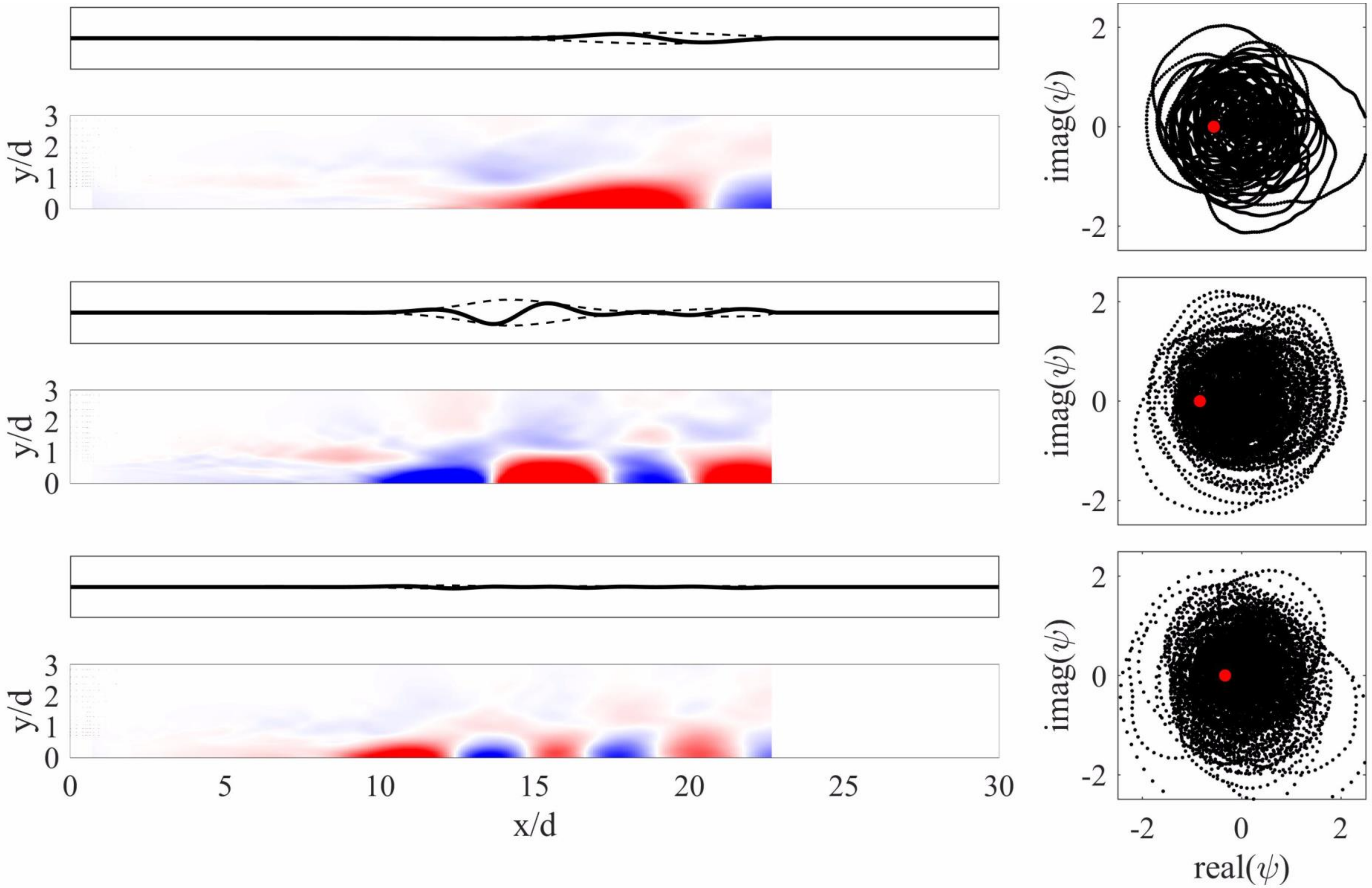
$$\boldsymbol{\Phi}(\boldsymbol{x})e^{-i\omega t}$$

- The real temporal behavior is more complicated…

HPOD results



The real temporal behavior



Wavepackets in jet flows: summary

- The **Hilbert POD** provides a complex-value extension of classic POD.
- A **space-only** implementation is obtained by switching from time to convective direction to provide the phase information of the wave.
- The **space-only HPOD** can extract **wavepackets from turbulent jet** flow fields without need of temporally resolved data.
- These wavepackets are not spectrally pure but live in a **time-frequency domain**.
- These velocity wavepackets can be used to decompose the Lighthill's analogy source term using a Galerkin projection…

… but this another story (for the *AIAA/CEAS Aeroacoustics 2024* in Rome).

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